Repeated Games through a Cyclic Operator: A Golden-Ratio Threshold, Fractal Payoff Sets, and the Golden-Window Conjecture

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** "Independent Researcher" is a provisional title used for clarity; in fact the author is an undergraduate at ZEN University and has no research affiliation.*
Version 2 changes only the author line and the wording of the declaration on the use of language models. The mathematical content is unchanged from version 1.

Abstract. In a discounted repeated game, the continuation values along a cyclic path of play are the fixed point of the linear operator $x \mapsto DQx + (I - D)p$ with $D = \delta I$ and Q a cyclic shift. We use this identity to study the symmetric n-player game in which a player whose k co-players cooperate earns $k + 1$ by cooperating and $k + 2$ by defecting. We prove that the three-player rotation scheme, in which exactly one player defects in each period, is a subgame-perfect equilibrium path if and only if $\delta \geq 1/\varphi \approx 0.618$, where φ is the golden ratio, and that the n-player threshold lies in $\big(1/(n-2) - 2/(n-2)^n,\ 1/(n-2)\big)$. For the payoff vectors $(2 + t, 4 - t, 2 + t)$, $0 < t \leq 1$, we determine exactly when they are equilibrium payoffs with public randomization, namely $\delta \geq \max(1/2, 1 - t)$, and show that the endpoint $(2, 4, 2)$ is never sustainable. Without public randomization the answer depends on δ in three regimes: two points for $1/2 \leq \delta < 1/\varphi$, a countably infinite set at $\delta = 1/\varphi$, and the full segment for $\delta \geq 2/3$. Inside the window $1/\varphi < \delta < 2/3$ the set of sustainable t has positive Hausdorff dimension, with an explicit lower bound. We conjecture that it has Lebesgue measure zero there and that its dimension increases continuously from 0 to 1 across the window. Numerical evidence is reported. Finally, in the linear dilemma with benefit b and cost c, the rotation is sustainable for some $\delta < 1$ exactly when cooperation is efficient, the three-player threshold is $(\sqrt{1 + 4c/b} - 1)/2$, and the rotation always needs more patience than cooperation by everyone. The literature has not been searched systematically, and some of the results may be known.

Keywords: repeated games; subgame-perfect equilibrium; rotation scheme; turn-taking; public randomization; golden ratio; fractal equilibrium payoffs; expansions in non-integer bases

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1. Introduction

Cooperation in a repeated social dilemma can be organised by taking turns. In a rotation scheme one player at a time enjoys the gain from defecting while the others cooperate, and the role passes around the group. Whether such a scheme is self-enforcing depends on how patient the players are. This

paper computes the exact patience threshold in a simple symmetric game and follows the same game to a set of payoffs whose structure is fractal.

The starting point is an identity. Along a path of play that repeats with period m, a player's continuation values satisfy $V = (1-\delta)p + \delta S V$, where p lists the stage payoffs by phase and S is the cyclic shift. The vector V is therefore the fixed point of the operator $x \mapsto DQx + (I - D)p$ with $D = \delta I$ and $Q = S$. The identity is the textbook formula for a discounted value. The operator is a special case of the Friedkin–Johnsen model of opinion dynamics (Friedkin and Johnsen 1990) with a cyclic influence matrix, and it was the object of the author's earlier preprints (Nemoto 2026a, 2026b). Those preprints placed the operator beside a three-player repeated game without connecting the two. Lemma 1 below connects them, and the subgame-perfection test becomes a componentwise inequality on the fixed point.

The main results are as follows.

1. The three-player rotation is a subgame-perfect equilibrium path if and only if $\delta \geq 1/\varphi$ (Proposition 1). For n players the threshold lies just below $1/(n-2)$ (Proposition 2).
2. The payoff vector $(2, 4, 2)$ is not an equilibrium payoff for any $\delta < 1$ (Proposition 3). This corrects an example in Nemoto (2026b).
3. With public randomization, $(2 + t, 4 - t, 2 + t)$ is an equilibrium payoff if and only if $\delta \geq \max(1/2, 1 - t)$ (Proposition 4).
4. Without public randomization the set T_δ of sustainable t passes through three regimes (Proposition 5), and inside the window $1/\varphi < \delta < 2/3$ it has positive Hausdorff dimension (Proposition 6).
5. The golden-window conjecture (Section 8) states that inside the window T_δ has Lebesgue measure zero and a dimension that increases continuously from 0 to 1.
6. In the linear dilemma with benefit b and cost c, the rotation is sustainable for some $\delta < 1$ exactly when $(n-1)b > c$, and it always needs a higher discount factor than cooperation by everyone (Proposition 7).

The same quadratic $\delta^2 + \delta - 1$ governs both the rotation threshold and the lower end of the window. Section 10 records this as an open question.

Scope. The author has not been able to search the literature systematically. Rotation schemes and fractal equilibrium payoff sets have both been studied (see Berg and Kitti 2014 for the latter), and some of the propositions below may be known. They are stated with complete proofs so that they can be checked. Every number in the paper is produced by a script that is publicly available (see *Data and code availability*).

2. The game

Fix $n \geq 3$. In the stage game G_n each of n players chooses C (cooperate) or D (defect). If k of a player's $n-1$ co-players choose C, the player's payoff is

$$\pi_C(k) = k + 1, \qquad \pi_D(k) = k + 2. \tag{1}$$

D strictly dominates C, all-D is the unique stage equilibrium, and it gives every player 2. Since D is a best reply to every profile, 2 is also each player's minmax payoff.

The stage game is repeated infinitely often with common discount factor $\delta \in (0,1)$ and perfect monitoring. Payoffs are normalised, $(1-\delta)\sum_{t\ge 0}\delta^t u_t$. Because the minmax payoff is a stage equilibrium payoff, permanent reversion to all-D is an optimal penal code (Abreu 1988). A path of play is therefore a subgame-perfect equilibrium path if and only if, in every period, every player's continuation value is at least

$$(1-\delta)\,d + 2\delta, \tag{2}$$

where d is the player's best one-period deviation payoff in that period.

3. Continuation values as a fixed point

Lemma 1. Let a path of play repeat with period m, and let p_k be a given player's stage payoff in phase k, $k = 0,\dots,m-1$. The player's continuation values V_k at the start of phase k satisfy

$$V = (1-\delta)(I-\delta S)^{-1}p, \qquad (SV)_k = V_{k+1 \bmod m}. \tag{3}$$

Equivalently, V is the unique fixed point of $x \mapsto DQx + (I-D)p$ with $D=\delta I$ and $Q = S$.

Proof. By definition $V_k = (1-\delta)p_k + \delta V_{k+1}$, that is $V = (1-\delta)p + \delta S V$. Since $\|\delta S\| = \delta < 1$, the matrix $I-\delta S$ is invertible. $\square$

For the three-player rotation of Section 4 with $\delta = 0.7$, formula (3) gives $V = (2.913242,\ 2.447489,\ 2.639269)$. A direct sum over 2000 periods agrees to within 10^{-14}.

4. Rotation thresholds

In the three-player rotation the profile cycles through $(D,C,C)\to(C,D,C)\to(C,C,D)\to\cdots$. A player's stage payoffs by phase are $p=(4,2,2)$, starting from the phase in which that player defects.

Proposition 1. In G_3 the rotation is a subgame-perfect equilibrium path if and only if

$$\delta \ge \frac{\sqrt{5}-1}{2} = \frac{1}{\varphi}. \tag{4}$$

Proof. The defector plays a best reply. A cooperator can switch to D and obtain 3 for one period, so by (2) the constraint is $V \ge 3(1-\delta)+2\delta = 3-\delta$. By Lemma 1 the smallest continuation value belongs to the player whose own defection is furthest away, and it equals $(2+2\delta+4\delta^2)/(1+\delta+\delta^2)$. The inequality

$$\frac{2+2\delta+4\delta^2}{1+\delta+\delta^2} \ge 3-\delta$$

is equivalent to $\delta^3+2\delta^2-1 = (\delta+1)(\delta^2+\delta-1)\ge 0$. The positive root of $\delta^2+\delta-1$ is $1/\varphi$. $\square$

Bisection on the inequalities of Lemma 1, without the polynomial, gives the threshold 0.618034, which agrees with $1/\varphi$ to six decimal places.

In the n-player rotation a single defector passes round the group. A cooperator earns $n-1$, the defector earns $n+1$, and a cooperator who deviates earns n.

Proposition 2. For each $n \geq 3$ there is a unique $\delta_n \in (0,1)$ such that the n-player rotation is a subgame-perfect equilibrium path if and only if $\delta \geq \delta_n$. Moreover

$$\frac{1}{n-2} - \frac{2}{(n-2)^n} < \delta_n < \frac{1}{n-2}. \tag{5}$$

Proof. A player who is j periods away from their own defection has continuation value $n-1+2h_j(\delta)$ with $h_j(\delta) = (1-\delta)\delta^j/(1-\delta^n)$. The constraint binds at $j = n-1$, so the rotation is an equilibrium path if and only if

$$f_n(\delta) = (n-2)\delta - 1 + 2h(\delta) \geq 0, \qquad h(\delta) = \frac{\delta^{n-1}}{1+\delta+\cdots+\delta^{n-1}}.$$

The function h is increasing, because $1/h(\delta) = \delta^{-(n-1)} + \cdots + \delta^{-1} + 1$ is a sum of decreasing terms. Hence f_n is strictly increasing, with $f_n(0) = -1$ and $f_n(1) = n-3+2/n > 0$, and its unique root is δ_n. Since $h > 0$, $\delta_n < 1/(n-2)$. Since $h(\delta) \leq \delta^{n-1}$, we get $1 - (n-2)\delta_n < 2(n-2)^{-(n-1)}$, which is the lower bound. $\square$

Table 1. Rotation thresholds δ_n, computed by bisection.

n	3	4	5	6	7	8	9	10
δ_n	0.618034	0.448223	0.328121	0.249636	0.199980	0.166666	0.142857	0.125000

By (5) the gap $1/(n-2) - \delta_n$ decays faster than any power of $1/n$.

5. A segment of payoff vectors

Nemoto (2026b, §4) describes the payoff vector $(2,4,2)$ of G_3 as sustainable at a sufficiently high discount factor. This is incorrect.

Proposition 3. For no $\delta < 1$ is $(2,4,2)$ an equilibrium payoff of the repeated G_3, whether subgame-perfect or Nash, and whether or not mixed strategies or public randomization are allowed.

Proof. The largest stage payoff in G_3 is 4, and only the profile (C,D,C) gives it to player 2. An average of 4 for player 2 therefore requires (C,D,C) in every period on the path, where player 1 earns 2. By switching to D, player 1 earns 3 at once and at least 2 in every later period, since D always guarantees 2. The deviation is worth at least $3(1-\delta) + 2\delta > 2$. $\square$

The vectors $(2+t, 4-t, 2+t)$ with $0 < t \leq 1$ form the segment from $(2,4,2)$ to $(3,3,3)$.

Proposition 4. With a public randomization device, $(2+t, 4-t, 2+t)$ is a subgame-perfect equilibrium payoff if and only if

$$\delta \geq \max\Big(\frac{1}{2},\ 1-t\Big). \tag{6}$$

Proof. Every stage payoff vector satisfies $v_2 + (v_1 + v_3)/2 \le 6$, with equality only at (C,C,C) and (C,D,C). The target attains 6, so the path uses only these two profiles. In both, player 1 cooperates and gains exactly 1 by deviating. By (2), player 1's continuation value must be at least

$$c = 2 + \frac{1-\delta}{\delta}. \tag{7}$$

Continuation values never exceed 3 on such a path, so $c \le 3$, which is $\delta \ge 1/2$. Player 1's value today is then at least $2(1-\delta) + \delta c = 3 - \delta$, which is $t \ge 1 - \delta$. The argument does not use purity, so it also covers mixed strategies.

Conversely, every value in $[3-\delta, 3]$ is a lottery between (C,C,C) followed by the value 3 and (C,D,C) followed by c. Both continuation values lie in the same interval, so the interval is self-generating (Abreu, Pearce and Stacchetti 1990). Player 2's constraint in (C,C,C) requires a continuation value of $6 - w \ge 3 \ge c$ for player 2, which holds. $\square$

At $t = 1$ the threshold is $1/2$. As $t \to 0$ it tends to 1, so $(2,4,2)$ is approached by sustainable payoffs but never reached.

6. Pure strategies without randomization

From now on we use pure strategies and no randomization device. As in the proof of Proposition 4, the path uses only (C,C,C) and (C,D,C), and player 1's continuation value w must remain in $[c,3]$. From one period to the next it moves by

$$w' = \frac{w - (1-\delta)u}{\delta}, \qquad u \in \{2,3\}, \tag{8}$$

which is the scalar recursion $w = (1-\delta)u + \delta w'$ read backwards. Let T_δ be the set of $t \in (0,1]$ for which $(2+t, 4-t, 2+t)$ is a pure-strategy subgame-perfect equilibrium payoff.

Proposition 5.

1. If $\delta < 1/2$, then $T_\delta = \varnothing$.
2. If $1/2 \le \delta < 1/\varphi$, then $T_\delta = \{\delta, 1\}$.
3. If $\delta = 1/\varphi$, then T_δ is countably infinite.
4. If $\delta \ge 2/3$, then $T_\delta = [1-\delta, 1]$.

Proof. Part 1 follows from Proposition 4. By (8), the branch $u = 2$ keeps w' in $[c,3]$ if and only if $w \le 2 + \delta$, and the branch $u = 3$ does so if and only if $w \ge 4 - 2\delta$. The first range meets $[c,3]$ if and only if $c \le 2 + \delta$, that is $\delta^2 + \delta - 1 \ge 0$, which is the polynomial of Proposition 1.

For $\delta < 1/\varphi$ only the branch $u = 3$ is available. It moves every $w < 3$ away from 3, so only $w = 3$ can be continued forever. The admissible starting values are $2 + \delta$ and 3, which gives part 2.

At $\delta = 1/\varphi$ we have $c = 2 + \delta$, so the branch $u = 2$ is available only at $w = c$, and it sends c to 3. Every other value is moved away from 3 by $u = 3$. The values that can be continued forever

are therefore 3 and the points $3-(3-c)\delta^k$, $k\ge 0$, which reach c after k periods. The admissible starting values are their images under the two branches, a countably infinite set. This gives part 3.

For $\delta \ge 2/3$ the two ranges cover $[c,3]$, so every value in it can be continued forever. From the starting values $[3-\delta,3]$ the two branches reach $[c,3]$, which gives part 4. $\square$

A direct search 40 periods ahead on the grid $t \in \{0.01, 0.02, \dots, 1\}$ finds $T_\delta \cap \text{grid} = \{0.5, 1\}$, $\{0.55, 1\}$ and $\{0.6, 1\}$ at $\delta = 0.5$, 0.55 and 0.6. At $\delta = 1/\varphi$, each of the first eight points $3-(3-c)\delta^k$ reaches 3.

7. Positive dimension inside the window

Change variables to $x = (3-w)/(1-\delta)$. The recursion $w = (1-\delta)u + \delta w'$ becomes

$$x = a + \delta x', \qquad a = 3-u \in \{0,1\}, \tag{9}$$

on the interval $[0, X]$ with $X = (2\delta-1)/\big(\delta(1-\delta)\big)$. Read backwards, $x' = (x-a)/\delta$. The branch $a=0$ keeps x' in $[0,X]$ if and only if $x \le \delta X$, and the branch $a=1$ does so if and only if $x \ge 1$. This is an expansion in the non-integer base $\beta = 1/\delta$ with the hole $(\delta X, 1)$. Inside the window, $\delta X < 1 < X$ and $3/2 < \beta < \varphi$.

Proposition 6. Let $1/\varphi < \delta < 2/3$. Let n_0 be the least integer $n \ge 0$ with $\delta^{-(n+1)} \ge X/(X-1)$, and let $s > 0$ be the unique solution of

$$\delta^{(n_0+1)s} = 1 - \delta^s. \tag{10}$$

Then the Hausdorff dimension of T_δ is at least s. In particular it is positive.

Proof. Take $x \in [1, X]$, apply the branch $a=1$ once and then $a=0$ for n periods. The result is $\delta^{-(n+1)}(x-1)$, which lies in $[1,X]$ if and only if

$$x \in J_n = \big[\, 1+\delta^{n+1},\; 1+X\delta^{n+1} \,\big].$$

Every intermediate value lies in $[0, \delta X]$, so the orbit never meets the hole. $J_n \subset [1,X]$ if and only if $\delta^{-(n+1)} \ge X/(X-1)$, that is $n \ge n_0$. The intervals J_n and J_{n+1} are disjoint because $X\delta < 1$. The composite branch maps J_n affinely onto $[1,X]$ with ratio $\delta^{-(n+1)}$.

For $N \ge n_0$, the inverse branches $[1,X] \to J_n$, $n_0 \le n \le N$, form a finite family of similarities with ratios δ^{n+1} and disjoint images. Their attractor K_N satisfies the open set condition, so its Hausdorff dimension s_N solves $\sum_{n=n_0}^{N} \delta^{(n+1)s_N} = 1$ (Hutchinson 1981). Every point of K_N has an orbit that stays out of the hole forever, so K_N corresponds to a set of continuation values that can be continued forever. As $N \to \infty$, s_N increases to the solution s of $\sum_{n \ge n_0} \delta^{(n+1)s} = 1$, which sums to (10). Finally, T_δ contains the preimage of this set under an affine bijection (one step of (8) followed by $t = w-2$), so $\dim_H T_\delta \ge s_N$ for every N, and hence $\dim_H T_\delta \ge s$. $\square$

The values of n_0 and s for five values of δ are the last two columns of Table 2.

8. The golden-window conjecture

Between $1/\varphi$ and $2/3$ both branches of (8) are available, but they leave the hole $(2+\delta,\ 4-2\delta)$ between their ranges. A value that falls into the hole cannot be continued, and the surviving values are those whose orbit never meets it. Below the window only two points survive, at $1/\varphi$ a countable set does, and above it the whole segment does (Proposition 5). Inside the window the set has positive dimension (Proposition 6).

Golden-window conjecture. For $1/\varphi < \delta < 2/3$:

(a) T_δ has Lebesgue measure zero;

(b) the Hausdorff dimension of T_δ is less than 1, is continuous and non-decreasing in δ, and tends to 0 as $\delta \downarrow 1/\varphi$ and to 1 as $\delta \uparrow 2/3$.

Numerical evidence. Starting from $[c,3]$, we apply the two inverse branches for 22 periods and keep the intervals that remain in $[c,3]$. Below $2/3$ the images of the two branches are disjoint, so the surviving intervals do not overlap. The growth rate is the geometric mean of the ratio of interval counts over the last five periods, and the dimension estimate is $\log(\text{growth})/\log(1/\delta)$.

Table 2. Surviving intervals after 22 periods, and the lower bound of Proposition 6.

δ	intervals	total length	growth	dimension estimate	n_0	lower bound s
0.62	134	5.16e-04	1.199	0.38	8	0.404
0.63	756	6.49e-03	1.325	0.61	4	0.609
0.64	1728	2.64e-02	1.380	0.72	3	0.722
0.65	3691	8.49e-02	1.427	0.83	3	0.748
0.66	5896	2.32e-01	1.466	0.92	2	0.920

In every row the growth rate is below $1/\delta$, so the total length keeps shrinking, in line with part (a). The dimension estimate increases with δ, in line with part (b). Neither observation is a proof. The estimates come from a finite depth, and the limits at the ends of the window are extrapolated. At $\delta = 0.62$ the finite-depth estimate 0.38 is below the proved lower bound 0.404, so the estimate is too low there.

Monotonicity is the least supported part of the conjecture. As δ changes, the base β and the hole move together, and dimension functions of maps with holes can behave like a devil's staircase. Five values of δ cannot settle this.

9. Rotation with a general benefit and cost

Replace the payoffs (1) by a linear social dilemma. A cooperator pays a cost $c>0$ and gives a benefit $b>0$ to every co-player, so that

$$\pi_C(k) = bk - c, \qquad \pi_D(k) = bk. \tag{11}$$

Up to the additive constant 2, the game G_n is the case $b = c = 1$. Write $r = c/b$. D is still a best reply to every profile, the minmax payoff is 0, and permanent reversion to all-D is still an optimal penal code. In biological terms δ can be read as the probability that the interaction continues.

Proposition 7. In the repeated game with stage payoffs (11):

1. The n-player rotation is a subgame-perfect equilibrium path for some $\delta < 1$ if and only if $r < n - 1$, that is, if and only if cooperation is efficient, $(n-1)b > c$. In that case there is a unique threshold $\delta_n(r)$, and
$$\frac{r}{n-2} - \frac{(1+r)\,r^{n-1}}{(n-2)^n} < \delta_n(r) < \frac{r}{n-2}. \tag{12}$$
2. For three players, $\delta_3(r) = \big(\sqrt{1+4r} - 1\big)/2$. At $r = 1$ this is $1/\varphi$.
3. $\delta_n(r) > r/(n-1)$, the threshold above which cooperation by everyone in every period is an equilibrium path.

Proof. The argument of Proposition 2 applies with the payoffs (11). The constraint binds for the cooperator whose own defection is furthest away, and the rotation is an equilibrium path if and only if
$$g(\delta) = (n-2)\delta - r + (1+r)\,h(\delta) \geq 0,$$
with h as in Proposition 2. The function g is strictly increasing, with $g(0) = -r$ and $g(1) = n - 2 - r + (1+r)/n$. The latter is positive if and only if $(n-1)^2 > r(n-1)$, that is $r < n - 1$. Since $h > 0$, $\delta_n(r) < r/(n-2)$. Since $h(\delta) < \delta^{n-1}$ for $0 < \delta < 1$, we get $r - (n-2)\delta_n(r) < (1+r)\big(r/(n-2)\big)^{n-1}$, which is the lower bound in (12). This proves part 1.

For $n = 3$, multiplying $g(\delta) \geq 0$ by $1 + \delta + \delta^2$ gives $(\delta+1)(\delta^2 + \delta - r) \geq 0$, which proves part 2.

Cooperation by everyone is an equilibrium path if and only if $b(n-1) - c \geq (1-\delta)\,b(n-1)$, that is $\delta \geq \delta^* = r/(n-1)$. At $\delta = \delta^*$ we have $r = (n-1)\delta^*$ and $g(\delta^*) = (1+r)h(\delta^*) - \delta^*$. Now $(1+r)(\delta^*)^{n-2} = (\delta^*)^{n-2} + (n-1)(\delta^*)^{n-1}$, which is less than $1 + \delta^* + \cdots + (\delta^*)^{n-1}$, because each of the $n-2$ terms $1, \delta^*, \ldots, (\delta^*)^{n-3}$ exceeds $(\delta^*)^{n-1}$. Hence $g(\delta^*) < 0$ and $\delta_n(r) > \delta^*$, which proves part 3. $\square$

Bisection on the inequalities of Lemma 1 gives $\delta_3(1/2) = 0.366025$ and $\delta_3(3/2) = 0.822876$, in agreement with the closed form. At $r = 1$ the rotation needs $\delta \geq 0.618034$, whereas cooperation by everyone needs only $\delta \geq 0.5$.

In this linear game, taking turns is never the easier way to cooperate. It needs more patience than cooperation by everyone, and its average payoff per player, $(n-1)\big(b(n-1) - c\big)/n$, is lower. The game therefore does not explain turn-taking. Turn-taking can pay only where benefits are not linear, for instance where one contributor suffices and a second is wasted. Such games are not studied here, and the literature on turn-taking has not been checked.

10. Discussion

Relation to existing work. Berg and Kitti (2014) show that pure-strategy equilibrium payoff sets of discounted repeated games can be fractals and measure them by Hausdorff dimension. Their methods may settle the conjecture for this game. The change of variables (9) places the problem among expansions in non-integer bases and expanding maps with holes, where part (a) and the limits in part (b) may follow from known results. Three leads are Glendinning and Sidorov (2001) on unique expansions in non-integer bases, Kalle, Kong, Langeveld and Li (2020) on the β-transformation with a hole, and Urbański (1986) on the dimension of sets that avoid a hole. The author has not checked whether these results apply.

Why $1/\varphi$ twice. The rotation threshold of Proposition 1 and the lower end of the window in Proposition 5 come from the same polynomial $\delta^2+\delta-1$. Algebraically the two conditions meet: when $\delta^2=1-\delta$, also $1-\delta^3=2\delta^2$, so the condition of Proposition 1, $2\delta^2\geq 1-\delta^3$, and that of Proposition 5, $\delta^2\geq 1-\delta$, change sign at the same point. At $\delta=1/\varphi$ both sides of the first equal 0.763932. The coincidence may depend on the payoffs (1). By Proposition 7 the rotation threshold in the linear dilemma is the root of $\delta^2+\delta-r$, so it moves with $r=c/b$. The analogue of the window for general b and c has not been computed, so it is not known whether the two thresholds separate. Whether the coincidence reflects a general link between rotation thresholds and the onset of branching in the value dynamics is left open.

What a proof would give. If general results such as those of Berg and Kitti (2014) apply directly, this game becomes a worked example of a known theory. If the conjecture requires a new argument, part (a) appears to be the more accessible half: it would follow from showing that the surviving length shrinks by a fixed factor over a bounded number of periods.

11. Conclusion

A single identity turns the continuation values of a cyclic path into the fixed point of a linear operator. In the game (1) this gives the exact threshold $1/\varphi$ for the three-player rotation, a correction of a published example, and an exact description of when a segment of payoff vectors is sustainable with public randomization. Without randomization the same segment passes through a window of discount factors in which the sustainable set is fractal, with a proved positive lower bound on its dimension. Whether that set has measure zero, and how its dimension depends on δ, is the golden-window conjecture. In the linear dilemma with general benefit and cost, the rotation is never easier to sustain than cooperation by everyone.

Data and code availability

Every number in this paper is produced by `games/games.py` in the repository https://github.com/cpsbvbng26-dotcom/trinity-operator (MIT licence). The script `verification/check_games.py` checks that every number in the accompanying note agrees with the script and verifies each proposition by an independent computation. The note itself is maintained at https://github.com/cpsbvbng26-dotcom/trinity-operator/tree/main/games.

Declaration on the use of language models

During the preparation of this work the author used Claude Code (Anthropic), a tool built on a large language model, to draft the text, to work out and write the propositions and their proofs, to formulate the conjecture, and to write the code that computes and checks every number. After using this tool, the author reviewed and edited the content as needed and takes full responsibility for the content of the publication. Claude is not an author.

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References

Abreu, D. (1988). On the theory of infinitely repeated games with discounting. *Econometrica*, 56(2), 383–396.

Abreu, D., Pearce, D., and Stacchetti, E. (1990). Toward a theory of discounted repeated games with imperfect monitoring. *Econometrica*, 58(5), 1041–1063.

Berg, K., and Kitti, M. (2014). Fractal geometry of equilibrium payoffs in discounted supergames. *Fractals*, 22(4).

Friedkin, N. E., and Johnsen, E. C. (1990). Social influence and opinions. *Journal of Mathematical Sociology*, 15(3–4), 193–206. https://doi.org/10.1080/0022250X.1990.9990069

Glendinning, P., and Sidorov, N. (2001). Unique representations of real numbers in non-integer bases. *Mathematical Research Letters*, 8(4), 535–543.

Hutchinson, J. E. (1981). Fractals and self similarity. *Indiana University Mathematics Journal*, 30(5), 713–747.

Kalle, C., Kong, D., Langeveld, N., and Li, W. (2020). The β-transformation with a hole at 0. *Ergodic Theory and Dynamical Systems*, 40(9), 2482–2514.

Nemoto, T. (2026a). The Trinity-Infinity Framework, Series I: A conceptual model with illustrative applications in game theory, mathematical logic, and mechanical engineering (revised). Zenodo. https://doi.org/10.5281/zenodo.22058624

Nemoto, T. (2026b). The Trinity-Infinity Framework, Series II: Extended formalization and further illustrative applications (revised). Zenodo. https://doi.org/10.5281/zenodo.22058777

Urbański, M. (1986). On Hausdorff dimension of invariant sets for expanding maps of a circle. *Ergodic Theory and Dynamical Systems*, 6(2), 295–309.